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A. Jimémez-Casa and A. Rodríguez Bernal

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http://www.mat.ucm.es/deptos/ma
e-mail:matemática_aplicada@mat.ucm.es

Dynamic boundary conditions as a singular limit of parabolic problems with terms concentrating at the boundary *

Ángela Jiménez-Casas[1]
Aníbal Rodríguez-Bernal[2]

1 Grupo de Dinámica No Lineal. Universidad Pontificia Comillas de Madrid. C/Alberto Aguilera 23, 28015 Madrid.

2 Departamento de Matemática Aplicada. Universidad Complutense de Madrid, 28040 Madrid and Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM.

Abstract

We obtain nonhomogeneous dynamic boundary conditions as a singular limit of a parabolic problems with null flux and potentials and reactions terms concentrating at the boundary.

Keywords: Dynamic boundary conditions, parabolic problems, concentrating integrals, singular perturbation.

1 Introduction.

Let Ω be an open bounded smooth set in $I\!R^N$ with a C^2 boundary $\Gamma = \partial\Omega$. Define the strip of width ε and base Γ as

$$\omega_\varepsilon = \{x - \sigma\vec{n}(x),\ x \in \Gamma,\ \sigma \in [0, \varepsilon)\}$$

for sufficiently small ε, say $0 \leq \varepsilon \leq \varepsilon_0$, where $\vec{n}(x)$ denotes the outward normal vector. We note that the set ω_ε is a neighborhood of Γ in $\bar{\Omega}$, that collapses to the boundary when the parameter ε goes to zero.

Then we consider the following family of parabolic problems

$$\left\{ \begin{array}{rll} \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t^\varepsilon - \Delta u^\varepsilon + \lambda u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x) u^\varepsilon & = f + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon & \text{in } \Omega \\ \frac{\partial u^\varepsilon}{\partial n} & = 0 & \text{on } \Gamma \\ u^\varepsilon(0, x) & = u_0^\varepsilon(x) & \text{in } \Omega \end{array} \right. \tag{1.1}$$

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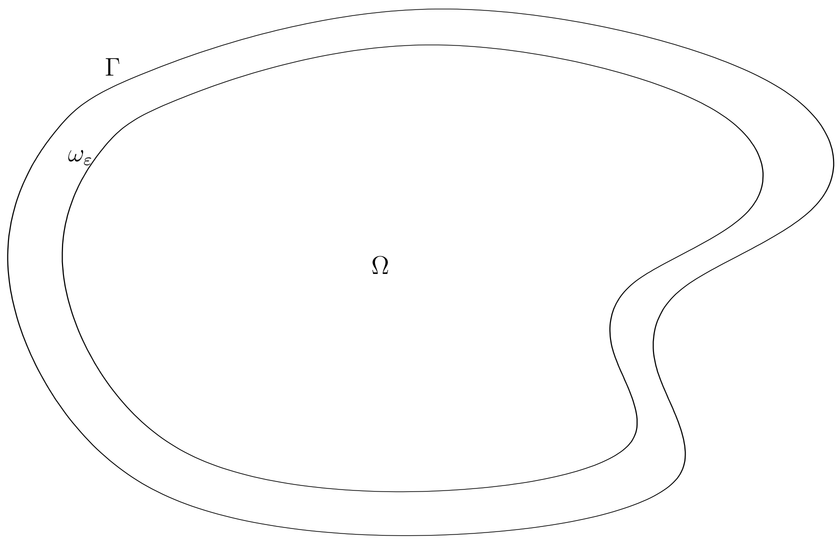

Figure 1: The set ω_ε

where $\mathcal{X}_{\omega_\varepsilon}$ is the characteristic function of the set ω_ε.

As ω_ε shrinks to the boundary as $\varepsilon \to 0$, the goal in this work is to show that dynamic boundary conditions can be obtained as a result of this limiting process. More precisely, the main result in this work is to prove that the family of solutions, u^ε, converges in some sense, when the parameter ε goes to zero, to a limit function u^0, which is given by the solution of the following parabolic problem with dynamic boundary conditions

$$\left\{ \begin{array}{rll} -\Delta u^0 + \lambda u^0 & = f & \text{in } \Omega \\ u^0_t + \frac{\partial u^0}{\partial n} + V(x) u^0 & = g & \text{on } \Gamma \\ u^0(0,x) & = v_0(x) & \text{on } \Gamma \end{array} \right. \tag{1.2}$$

where v_0, V and g are obtained as the limits of the concentrating terms

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon_0 \to v_0, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \tag{1.3}$$

in some sense that we make precise below. In particular, we will obtain that the time derivative of the solution concentrates to the time derivative of the restriction to the boundary, as $\varepsilon \to 0$.

Note that (1.1) is formally equivalent to solving

$$\left\{ \begin{array}{rll} -\Delta u^\varepsilon + \lambda u^\varepsilon & = f & \text{in } \Omega \setminus \bar{\omega}_\varepsilon \\ \frac{1}{\varepsilon} u^\varepsilon_t - \Delta u^\varepsilon + \lambda u^\varepsilon + \frac{1}{\varepsilon} V_\varepsilon u^\varepsilon & = f + \frac{1}{\varepsilon} g_\varepsilon & \text{in } \omega_\varepsilon \\ \frac{\partial u^\varepsilon}{\partial n} & = 0 & \text{on } \Gamma \\ u^\varepsilon(0,x) & = u^\varepsilon_0(x) & \text{in } \Omega \end{array} \right. \tag{1.4}$$

and that in (1.4) boundary conditions are missing on $\Gamma_\varepsilon = \partial\omega_\varepsilon \setminus \Gamma = \partial(\Omega \setminus \bar{\omega}_\varepsilon)$. Since there would be several ways of connecting the solutions of the elliptic and the parabolic equations in (1.4) along that boundary, we consider the boundary conditions on Γ_ε that ensure maximal smoothness of solutions. This is achieved by imposing the classical transmissions conditions on Γ_ε, that is, no jump of the u^ε and its normal derivate across Γ_ε, see [7],

$$[u_\varepsilon]_{\Gamma_\varepsilon} = [\frac{\partial u_\varepsilon}{\partial n}]_{\Gamma_\varepsilon} = 0. \tag{1.5}$$

Hence, (1.4) and (1.5) is a formulation of an elliptic–parabolic transmission problem, see [5], Chapter 1, Section 9, for related problems. The well posedness of (1.1), in the sense of (1.4), (1.5), will be addressed in Section 2.1 following the techniques in [7].

On the other hand, (1.2) must be understood as an evolution problem on the boundary Γ, such that, for each time $t > 0$, the solution must be lifted to the interior of Ω by means of the elliptic equation in (1.2). In this way the term $\frac{\partial u^0}{\partial n}$ becomes a linear nonlocal operator for functions defined on Γ. The well posedness of (1.2) will be discussed in Section 2.2 following the techniques in [6].

As for (1.3) the starting point are the results in [2] which state that if we consider a family of functions in Ω such that for some $p > 1$

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |h_\varepsilon|^p \le C$$

then, taking subsequences if necessary, one can assume that there exists $h_0 \in L^p(\Gamma)$ such that for any smooth function φ, defined in $\bar{\Omega}$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} h_\varepsilon\varphi = \int_\Gamma h_0\varphi.$$

In other words, the results above indicate that concentrating integrals near the boundary behave as boundary integrals and the concentrating functions behave as traces. Several results of this type for functions that also depend on time, will be obtained in Section 3. These results will be used then in Section 4 when proving that actually solutions of (1.1) converge to solutions of (1.2); see Proposition 4.2 and Theorem 4.6 which are the two main results concerning convergence of solutions. It is worth noting that for the linear potentials V_ε we will require the uniform integrability condition above for $p = \rho > N - 1$. In fact, for $\varepsilon > 0$ fixed, only $\rho > N/2$ is required for the elliptic part of the equation to be well defined. However for dealing with that family of problems, uniformly in ε, we need $\rho > N - 1$, since in the limit the interior potential behaves as a boundary potential which requires this sort of integrability. Indeed for part of the stronger convergence result in Theorem 4.6 we will actually require $\rho > 2(N - 1)$.

Finally, note that we will require $\lambda > \lambda_0$ for some $\lambda_0 > 0$ independent of ε. This will be necessary for the elliptic operators in (1.1) and (1.2) to be uniformly coercive in ε; see Lemma 3.5. This is due to the fact that the singular perturbation in (1.1) affects the time derivative of the unknown. In fact, if we replace in (1.1) the term $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t^\varepsilon$ by u_t^ε, that is, no concentration of the time derivative, then the usual change of variable $v(t) = e^{\alpha t}u(t)$, with a suitable $\alpha \in I\!R$, would allow to handle the problem for any $\lambda \in I\!R$. In this case this change of variables does not give the same result as the extra linear term introduced in the equation is concentrated near the boundary and then can not give uniform coercitivity uniformly in ε.

Note that related problems have been considered before. As mentioned above, [2] considered linear elliptic problems with concentrating terms near the boundary. Also [4] considered nonlinear parabolic problems with linear and nonlinear terms concentrating near the boundary and analyzed the proximity of the long time behavior of solutions by studying the proximity of the the corresponding global attractors. In both [4] and in this paper the results in [2] provide some of the building blocks of the analysis. Note however that the case considered here is more singular than the ones in the references quoted above, because the singular limit affects the time derivative of the solution.

As noted in [2], in the context of elliptic problems, the convergence results obtained below, despite its intrinsic mathematical interest, have potential applications in developing approximation schemes for (1.2). Numerical solutions of (1.1) can be obtained by suitable spectral or finite element methods. In both cases the setting gets rid of the zero flux condition. In fact, (1.1) has a natural and simple variational formulation not involving surface integrals or traces in Γ. On the other hand, solving (1.2) requires to use suitable sets of functions defined on the boundary, whose trace evolves according to the second equation in (1.2). This approach becomes more subtle if the boundary of the domain is not smooth enough.

2 On the well posedness of the approximating and limit problems

In this section we describe the well posedness results for (1.4) and (1.2). For this we will make use of minor variations of the results in [6, 7].

Here and below $H^{s,q}(\Omega)$ denote the Bessel potentials spaces which, for integer s, coincide with the usual Sobolev spaces; see [1]. In particular, for $q = 2$ these spaces will be simply denoted as $H^s(\Omega)$. Also, note that for $s > 0$ we denote

$$H^{-s,q}(\Omega) = (H^{s,q'}(\Omega))'.$$

In particular,when $q = 2$ we have the nonstandard notation $H^{-1}(\Omega) = (H^1(\Omega))'$. The same applies for spaces of traces. For example $H^{-1/2}(\Gamma)$ will denote the dual space of $H^{1/2}(\Gamma)$.

Also, we will consider below traces on Γ of functions defined in Ω. Hence, we will denote either by $\gamma(u)$ or by $u_{|\Gamma}$ the trace of a function u.

2.1 Well possedness of (1.4)

Note that in [7] it was considered a very similar problem to (1.4). In fact in [7] Dirichlet boundary conditions were assumed on Γ instead as Neumann ones as in this paper. Also it was assumed $V_\varepsilon = 0$. Therefore, we explain below how to modify the arguments in [7] to apply them to (1.4). See Theorem 1.1, Theorem 4.9 and Proposition 4.10 in [7].

Hence, we consider (1.4). Since $\varepsilon > 0$ is fixed, and in order to simplify the notations, we do not make explicit the dependence on ε.

We denote by $H^{-1}(\Omega)$ the dual space of $H^1(\Omega)$ and then

$$H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega).$$

Also, we define the bilinear symmetric form in $H^1(\Omega)$

$$a(\varphi, \phi) = \int_\Omega \nabla\varphi\nabla\phi + \int_\omega V\varphi\phi + \lambda\int_\Omega \varphi\phi \tag{2.1}$$

for every $\varphi, \phi \in H^1(\Omega)$. This bilinear form is coercive (and hence an equivalent scalar product in $H^1(\Omega)$) if λ is sufficiently large, provided

$$V \in L^\rho(\omega), \quad \rho > N/2. \tag{2.2}$$

In such a case, the bilinear form defines an isomorphism, L, between $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$ such that for every $\varphi, \phi \in H^1(\Omega)$

$$\Big\langle L(\varphi), \phi\Big\rangle_{-1,1} = \int_\Omega \nabla\varphi\nabla\phi + \int_\omega V\varphi\phi + \lambda\int_\Omega \varphi\phi.$$

Note that if $f \in H^{-1}(\Omega)$, the solution $u \in H^1(\Omega)$ of $L(u) = f$ is a weak solution of

$$\begin{cases} -\Delta u + \mathcal{X}_\omega V u + \lambda u & = f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} & = 0 \quad \text{on } \Gamma\ . \end{cases}$$

If $f \in L^2(\Omega)$ then $u \in H^2(\Omega)$ and $\frac{\partial u}{\partial n} = 0$ actually holds on the boundary.

We also identify $L^2(\omega)$ with its dual and consider the bilinear form restricted to $H^1(\omega)$, and the corresponding isomorphism L_ω between $H^1(\omega)$ and $H^{-1}(\omega)$.

Definition 2.1 *i) The set $Z(\Omega \setminus \bar\omega)$ is the orthogonal set to $H_0^1(\Omega \setminus \bar\omega)$ in $H^1(\Omega)$ with respect to the scalar product (2.1). That is, $u \in Z(\Omega \setminus \bar\omega)$ iff $u \in H^1(\Omega)$ and*

$$\Big\langle L(u), \phi\Big\rangle_{-1,1} = \int_\Omega \nabla u\nabla\phi + \lambda\int_\Omega u\phi = 0$$

for every $\phi \in H_0^1(\Omega \setminus \bar\omega)$, i.e. $L(u)_{|H_0^1(\Omega\setminus\bar\omega)} = 0$. In particular $-\Delta u + \lambda u = 0$ in the sense of distributions in $\Omega \setminus \bar\omega$.
ii) Denote $\Gamma_* = \partial\omega \setminus \Gamma = \partial(\Omega \setminus \bar\omega)$. Then for a given function $u \in H^1(\omega)$, we define the "$Z(\Omega \setminus \bar\omega)$ lifting" of u to $\Omega \setminus \bar\omega$, $v = Z(u) \in H^1(\Omega \setminus \bar\omega)$, as the solution of

$$\begin{cases} -\Delta v + \lambda v & = 0 \quad \textit{in } \Omega \setminus \bar\omega \\ v & = u \quad \textit{on } \Gamma_* \end{cases}$$

in the sense that

$$\int_{\Omega\setminus\bar\omega} \nabla v\nabla\phi + \lambda\int_{\Omega\setminus\bar\omega} v\phi = 0$$

for every $\phi \in H_0^1(\Omega \setminus \bar{\omega})$ and v satisfies the boundary data on Γ_*.

We also define

$$B(u) = \begin{cases} Z(u) & \text{in } \Omega \setminus \bar{\omega} \\ u & \text{in } \omega \ . \end{cases}$$

Therefore, $B(u) \in Z(\Omega \setminus \bar{\omega})$ and defines a linear mapping between $H^1(\omega)$ and $H^1(\Omega)$.
iii) For functions defined on Ω we define the "restriction" operator to ω by $R(u) = \mathcal{X}_\omega u$.

Then we have the following result, which is similar to Proposition 3.2 in [7].

Proposition 2.2 *i) We have the orthogonal decomposition $H^1(\Omega) = Z(\Omega \setminus \bar{\omega}) \bigoplus H_0^1(\Omega \setminus \bar{\omega})$ and each $u \in H^1(\Omega)$ can be split accordingly as $u = u_1 + u_2$ where $u_1 = B(R(u)) \in Z(\Omega \setminus \bar{\omega})$, $u_2 = u - B(R(u)) \in H_0^1(\Omega \setminus \bar{\omega})$.*

Moreover, $B : H^1(\omega) \mapsto Z(\Omega \setminus \bar{\omega})$ is an isomorphism, whose inverse is given by the operator R.
ii) Acting by restriction, we have the decomposition $H^{-1}(\Omega) = H^{-1}(\omega) \bigoplus H_0^{-1}(\Omega \setminus \bar{\omega})$.
iii) The operators

$$L_{\Omega \setminus \bar{\omega}} = L \mid_{H_0^1(\Omega \setminus \bar{\omega})} : H_0^1(\Omega \setminus \bar{\omega}) \mapsto H_0^{-1}(\Omega \setminus \bar{\omega})$$

and

$$A = LB : H^1(\omega) \mapsto H^{-1}(\omega)$$

are isomorphisms.

Therefore, given $h \in H^{-1}(\Omega)$, then $u \in H^1(\Omega)$ satisfies $L(u) = h$ iff $u = u_1 + u_2 = B(R(u)) + u_2$, as in i), with

$$R(u) = A^{-1}(h_1), \qquad u_2 = D(h_2)$$

where $D := L^{-1}_{\Omega \setminus \bar{\omega}}$ and $h = h_1 + h_2$ as in ii).

The isomorphism $A = LB : H^1(\omega) \mapsto H^{-1}(\omega)$ is given by

$$A = LB = L_\omega - \Big(\frac{\partial Z}{\partial n_*}\Big)_{\Gamma_*}$$

in the sense that for every $u, v \in H^1(\omega)$ one has

$$\Big\langle A(u), v \Big\rangle_{-1,1}^{\omega} = \int_\omega \nabla u \nabla v + \int_\omega Vuv + \lambda \int_\omega uv - \int_{\Gamma_*} \frac{\partial Z(u)}{\partial n_*} Z(v) =$$

$$= \int_\Omega \nabla B(u) \nabla B(v) + \int_\omega V B(u) B(v) + \lambda \int_\Omega B(u) B(v)$$

where n_* denotes the outward unit normal to ω along Γ_*. $\square$

Under the above notations, observe that for solving

$$\begin{cases} \mathcal{X}_\omega u_t - \Delta u + \mathcal{X}_\omega V u + \lambda u & = h \qquad \text{in } \Omega \\ \frac{\partial u}{\partial n} & = 0 \qquad \text{on } \Gamma \\ u(0,x) & = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{2.3}$$

if we assume that for each $t > 0$ we have $u(t) \in H^1(\Omega)$ and using the decomposition in Proposition 2.2, we must have

$$u(t) = B(R(u(t))) + D(h(t)) \quad \text{in} \quad \Omega.$$

Also the smooth matching across Γ_*, (1.5), now reads

$$\frac{\partial u}{\partial n_*} = \frac{\partial Z(u)}{\partial n_*} + \frac{\partial D(h)}{\partial n_*} \quad \text{on} \quad \Gamma_*, \tag{2.4}$$

where n_* denotes the outward unit normal to ω along Γ_*. Finally, note that for (1.4) we take $h = f + \mathcal{X}_\omega g$.

Therefore, in view of the properties of the operator A in Proposition 2.2,to solve (2.3) we are lead to solve an evolution problem of the form

$$\left\{ \begin{array}{ll} u(t) = B(v(t)) + D(h(t)) & \text{in } \Omega \\ v_t + Av = h_\omega + (\frac{\partial D(h)}{\partial n_*})_{\Gamma_*} & \text{in } \omega \end{array} \right. \tag{2.5}$$

assumed, that $h(t) \in L^2(\Omega)$. Note that in (2.5) we have reduced (2.3) and (2.4) to a nonhomogeneous evolution problem in ω with a well behaved operator A.

Also from Lemma 3.4 in [7], if $h \in L^2(\Omega)$ then its decomposition in $H^{-1}(\Omega)$ in Proposition 2.2 is given by $h_1 = h_\omega + \left(\frac{\partial D(h)}{\partial n_*}\right)_{|\Gamma_*} \in L^2(\omega) + H^{1/2}(\Gamma_*)$ and $h_2 = h_{\Omega \setminus \overline{\omega}} \in L^2(\Omega \setminus \overline{\omega})$. So, we have

Definition 2.3 *Assume $h(t) = h_1(t) + h_2(t) \in H^{-1}(\Omega)$ is given a.e. $t \in (0, T)$, with $h_1 \in H^{-1}(\omega)$ and $h_2 \in H_0^{-1}(\Omega \setminus \overline{\omega})$. Then a solution of (2.5), is a function $t \mapsto u(t) \in H^1(\Omega)$ such that for $t \in (0, T)$*

$$u(t) = B(v(t)) + D(h_2(t)) \in H^1(\Omega) \tag{2.6}$$

and $v(t) = Ru(t) \in H^1(\omega)$ satisfies

$$v(t) = e^{-At}v_0 + \int_0^t e^{-A(t-s)} h_1(s)\, ds \tag{2.7}$$

where v_0 is given in ω and e^{-At} is the semigroup generated by $-A$.

Note that with this Definition, the mild solution of (2.5) is explicitly given by (2.6)–(2.7) and it is therefore unique.

Then in a similar fashion as in Theorem 1.1 in [7] (see also Theorem 4.9 in that reference), we have the following result that states that the unique mild solution of (2.5) as in Definition 2.3 actually satisfies (2.3).

Theorem 2.4 *Assume h is given such that either*

a) $h \in W^{1,1}((0, T), L^2(\Omega))$ or

b) $h \in L^2((0, T), L^2(\omega)) = L^2((0, T) \times \omega)$ and $h \in W^{1,1}((0, T), L^2(\Omega \setminus \bar{\omega}))$.

Assume also $u_0 \in H^1(\Omega)$ satisfies

$$-\Delta u_0 + \lambda u_0 = h(0) \quad \text{in } \Omega \setminus \bar{\omega}. \tag{2.8}$$

Then the unique solution of (2.5), as in Definition 2.3, satisfies

$$u \in C([0,T), H^1(\Omega)) \cap L^2((0,T), H^2(\Omega)), \quad u_t \in L^2((0,T) \times \omega), \quad u(0) = u_0$$

and satisfies (2.3) in the sense that

$$\mathcal{X}_\omega u_t + L(u) = h \quad \text{in} \quad H^{-1}(\Omega), \quad \text{a.e. } t \in (0,T).$$

In particular, $u(t)$ satisfies (2.4) a.e. $t \in (0,T)$.

Also, as in Proposition 4.10 in [7], we get

Proposition 2.5 *Assume, as above, that $u_0 \in H^1(\Omega)$ satisfying (2.8) and $h(t) \in L^2(\Omega)$ a.e. $t \in (0,T)$, are given.*
i) If $h \in W^{1,1}((0,T), L^2(\Omega))$, then

$$\begin{aligned}\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_\omega V u(t)^2 \quad &+ \quad \lambda \|u(t)\|^2_{L^2(\Omega)} + 2\int_0^t \int_\omega u_t^2 = \|\nabla u_0\|^2_{L^2(\Omega)} + \int_\omega V u_0^2 + \lambda \|u_0\|^2_{L^2(\Omega)} + \\ &+2\left(\int_\Omega h(t)u(t) - \int_\Omega h(0)u_0 - \int_0^t \int_\Omega h_t u\right). \end{aligned} \tag{2.9}$$

Therefore, the mapping $(u_0, h) \longmapsto (u, u_t)$ is Lipschitz from $H^1(\Omega) \times W^{1,1}((0,T), L^2(\Omega))$ into $C([0,T], H^1(\Omega)) \times L^2((0,T) \times \omega)$.
ii) If $h \in L^2((0,T) \times \omega)$ and $h \in W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$, then

$$\begin{aligned}\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_\omega V u(t)^2 + \lambda \|u(t)\|^2_{L^2(\Omega)} + 2\int_0^t \int_\omega u_t^2 = \|\nabla u_0\|^2_{L^2(\Omega)} + \int_\omega V u_0^2 + \lambda \|u_0\|^2_{L^2(\Omega)} + \\ 2\left(\int_0^t \int_\omega h u_t + \int_{\Omega\setminus\overline{\omega}} h(t)u(t) - \int_{\Omega\setminus\overline{\omega}} h(0)u_0 - \int_0^t \int_{\Omega\setminus\overline{\omega}} h_t u\right). \end{aligned} \tag{2.10}$$

Therefore, the mapping $(u_0, h_\omega, h_{\Omega\setminus\overline{\omega}}) \longmapsto (u, u_t)$ is Lipschitz from $H^1(\Omega) \times L^2((0,T) \times \omega) \times W^{1,1}((0,T), L^2(\Omega \setminus \overline{\omega}))$ into $C([0,T], H^1(\Omega)) \times L^2((0,T) \times \omega)$.

2.2 Well possedness of (1.2)

We consider the parabolic problem (1.2), that is

$$\left\{ \begin{array}{lll} -\Delta u^0 + \lambda u^0 & = f & \text{in } \Omega \\ u^0_t + \frac{\partial u^0}{\partial n} + V_0(x) u^0 & = g & \text{on } \Gamma \\ u^0(0,x) & = v_0(x) & \text{on } \Gamma \end{array} \right. \tag{2.11}$$

for which we adapt the results in [6]. Note that the setting for this problem is pretty much in the spirit of the previous section, and therefore, we point out the main differences. The reader is then referred to [6] for full details.

In this case we define the bilinear symmetric form in $H^1(\Omega)$

$$a_0(\varphi,\phi) = \int_\Omega \nabla\varphi\nabla\phi + \int_\Gamma V_0\varphi\phi + \lambda\int_\Omega \varphi\phi \tag{2.12}$$

for every $\varphi,\phi \in H^1(\Omega)$.

Assuming

$$V_0 \in L^\rho(\Gamma), \quad \rho > N-1.$$

then for sufficiently large λ this bilinear form is coercive and hence an equivalent scalar product in $H^1(\Omega)$ and defines an isomorphism, L_0, between $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$ such that for every $\varphi,\phi \in H^1(\Omega)$

$$\left\langle L_0(\varphi),\phi\right\rangle_{-1,1} = \int_\Omega \nabla\varphi\nabla\phi + \int_\Gamma V_0\varphi\phi + \lambda\int_\Omega \varphi\phi.$$

Definition 2.6 *i) The set $Z_0(\Omega)$ is the orthogonal set to $H^1_0(\Omega)$ in $H^1(\Omega)$ with respect to the scalar product (2.12). That is, $u \in Z_0(\Omega)$ iff $u \in H^1(\Omega)$ and*

$$\left\langle L_0(u),\phi\right\rangle_{-1,1} = \int_\Omega \nabla u\nabla\phi + \lambda\int_\Omega u\phi = 0$$

for every $\phi \in H^1_0(\Omega)$, i.e. $L_0(u)_{|H^1_0(\Omega)} = 0$. In particular $-\Delta u + \lambda u = 0$ in the sense of distributions in Ω.
ii) For a given function u defined on Γ, we defined the "$Z_0(\Omega)$ lifting" of u to Ω, $v = B_0(u) \in H^1(\Omega)$, as the solution of

$$\begin{cases} -\Delta v + \lambda v & = 0 \quad \text{in } \Omega \\ \qquad v & = u \quad \text{on } \Gamma \end{cases}$$

in the sense that

$$\int_\Omega \nabla v\nabla\phi + \lambda\int_\Omega v\phi = 0$$

for every $\phi \in H^1_0(\Omega)$ and v satisfies the boundary data on Γ.

Then we have the following result, which is taken from Proposition 1.1 in [6] and is similar to Proposition 2.2 above. Note that here we denote by γ the trace operator in Γ.

Proposition 2.7 *i) We have the orthogonal decomposition $H^1(\Omega) = Z_0(\Omega) \bigoplus H^1_0(\Omega)$ and each $u \in H^1(\Omega)$ can be split accordingly as $u = u_1 + u_2$ where $u_1 = B_0(\gamma(u)) \in Z_0(\Omega)$, $u_2 = u - B_0(\gamma(u)) \in H^1_0(\Omega)$.*

Moreover, $B_0 : H^{1/2}(\Gamma) \mapsto Z_0(\Omega)$ is an isomorphism, whose inverse is given by the operator γ.
ii) Acting by restriction, we have the decomposition $H^{-1}(\Omega) = H^{-1/2}(\Gamma) \bigoplus H^{-1}_0(\Omega)$.
iii) The operators

$$L_\Omega = L_0 \,|_{H^1_0(\Omega)} : H^1_0(\Omega) \mapsto H^{-1}_0(\Omega)$$

and

$$A_0 = L_0 B_0 : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$$

are isomorphisms.

Therefore, given $h \in H^{-1}(\Omega)$*, then* $u \in H^1(\Omega)$ *satisfies* $L_0(u) = h$ *iff* $u = u_1 + u_2 = B_0(\gamma(u)) + u_2$*, as in i), with*

$$\gamma(u) = A_0^{-1}(h_1), \qquad u_2 = D_0(h_2)$$

where $D_0 := L_\Omega^{-1}$ *and* $h = h_1 + h_2$ *as in ii).*

The isomorphism $A_0 = L_0 B_0 : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$ *is given by*

$$A_0 = L_0 B_0 = \frac{\partial B_0}{\partial n} + V_0 I$$

and for every $u, v \in H^{1/2}(\Gamma)$ *one has*

$$\Big\langle A_0(u), v \Big\rangle^{\Gamma}_{-1/2,1/2} = \int_\Omega \nabla B_0(u) \nabla B_0(v) + \int_\Gamma V_0 uv + \lambda \int_\Omega B_0(u) B_0(v). \square$$

With this, solving (2.11) is equivalent to solving

$$\begin{cases} u^0(t) = B_0(\gamma(u(t)) + D_0(f(t)) & \text{in } \Omega \\ u^0_t + \frac{\partial B_0(u^0(t))}{\partial n} + V_0 u^0 = g - \frac{\partial D_0(f(t))}{\partial n} & \text{on } \Gamma \\ u^0(0) = v_0 & \text{on } \Gamma \end{cases} \tag{2.13}$$

assuming $f(t) \in L^2(\Omega)$.

As proved in Lemma 1.1 in [6], if $h = f_\Omega + g_\Gamma \in H^{-1}(\Omega)$ in the sense that

$$\Big\langle h, \phi \Big\rangle_{-1,1} = \int_\Omega f\phi + \int_\Gamma g\phi$$

for every $\phi \in H^1(\Omega)$, then the splitting $h = h_1 + h_2$ in Proposition 2.7 is given by $h_1 = g - \frac{\partial D_0(f)}{\partial n} \in H^{-\frac{1}{2}}(\Gamma)$ and $h_2 = f \in H_0^{-1}(\Omega)$. Hence, we have

Definition 2.8 *Assume* $h(t) = h_1(t) + h_2(t) \in H^{-1}(\Omega)$ *is given a.e.* $t \in (0, T)$*, with* $h_1(t) \in H^{-\frac{1}{2}}(\Gamma)$ *and* $h_2(t) \in H_0^{-1}(\Omega)$*. Then the solution of (2.13) is a function* $t \mapsto u^0(t) \in H^1(\Omega)$ *such that for* $t \in (0, T)$

$$u^0(t) = B_0(v(t)) + D_0(h_2(t)) \in H^1(\Omega) \tag{2.14}$$

and $v(t) = \gamma(u^0(t)) \in H^{\frac{1}{2}}(\Gamma)$ *satisfies*

$$v(t) = e^{-A_0 t} v_0 + \int_0^t e^{-A_0(t-s)} h_1(s) ds \tag{2.15}$$

where $e^{-A_0 t}$ *is the semigroup generated by* $-A_0$*.*

Note that with this Definition, the mild solution of (2.13) is explicitly given by (2.14)–(2.15) and it is therefore unique.

Now as in Corollary 3.3 in [6] we have the following result that states that the unique mild solution of (2.13) as in Definition 2.8 actually satisfies (2.11).

Proposition 2.9 *Assume $h(t) = h_1(t) + h_2(t) \in H^{-1}(\Omega)$ is given a.e. $t \in (0,T)$, with $h_1 \in L^2((0,T) \times \Gamma)$ and $h_2 \in L^2((0,T), H_0^{-1}(\Omega))$. Assume also $u_0 \in H^1(\Omega)$ is given.*

Then u^0 given by (2.14) and (2.15), with $v_0 = \gamma(u_0)$ satisfies

$$u^0 \in L^2((0,T), H^1(\Omega)), \quad \gamma(u^0)_t \in L^2((0,T) \times \Gamma)$$

$$\gamma(u^0)_t + L_0(u^0) = h \tag{2.16}$$

as an equality in $H^{-1}(\Omega)$, a.e. $t \in (0,T)$. In particular $\gamma(u^0) \in C([0,T], L^2(\Gamma))$.

Moreover, if $h_2 \in C([0,T), H_0^{-1}(\Omega))$ and u_0 satisfies

$$-\Delta u_0 + \lambda u_0 = h_2(0) \tag{2.17}$$

in Ω, i.e. $u_0 = B_0(\gamma(u_0)) + D_0(h_2(0))$, then

$$u^0 \in C([0,T), H^1(\Omega)), \quad u^0(0) = u_0. \tag{2.18}$$

In particular, the above applies if $h(t) = f_\Omega(t) + g_\Gamma(t)$, with $f(t) \in L^2(\Omega)$ and $g(t) \in H^{-1/2}(\Gamma)$ a.e. $t \in (0,T)$, $h_1 = g - \frac{\partial D_0(f)}{\partial \vec{n}} \in L^2((0,T) \times \Gamma)$ and $h_2 = f \in L^2((0,T), H_0^{-1}(\Omega))$. Finally (2.18) holds provided $f \in C([0,T), H_0^{-1}(\Omega))$ and (2.17).

3 Concentrating integrals

In this section we show several results that describe how different concentrated integrals converge to surface integrals. Hereafter we denote by $C > 0$ any positive constant such that C is independent of ε and t. This constant may change from line to line.

The following lemma is proved in [2] and basically states that concentrated functions behave as traces.

Lemma 3.1 *A) Assume that $v \in H^{s,p}(\Omega)$ with $\frac{1}{p} < s$ and such that $H^{s,p}(\Omega) \subset L^q(\Gamma)$, i.e. $s - \frac{N}{p} \geq -\frac{(N-1)}{q}$, or $v \in H^{1,1}(\Omega)$, i.e, $s = 1 = p$ and $q = 1$ below. Then for sufficiently small ε_0, we have, for some positive constant C independent of ε,*

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^q \leq C \|v\|^q_{H^{s,p}(\Omega)} \tag{3.1}$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^q = \int_\Gamma |v|^q. \tag{3.2}$$

B) Consider a family f_ε defined on ω_ε, such that for some $1 \leq r < \infty$ and a positive constant C independent of ε,

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |f_\varepsilon|^r \leq C.$$

Then, for every sequence converging to zero (that we still denote $\varepsilon \to 0$) there exists a subsequence (that we still denote the same) and a function $f_0 \in L^r(\Gamma)$ (or a bounded Radon measure on Γ, $f_0 \in \mathcal{M}(\Gamma)$ if $r = 1$) such that, for every $s > \frac{1}{p}$ and $H^{s,p}(\Omega) \subset L^{r'}(\Gamma)$ that is

$$s - \frac{N}{p} > -\frac{N-1}{r'} \tag{3.3}$$

we have that

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} f_\varepsilon \to f_0 \quad \text{in } H^{-s,p'}(\Omega) \text{ as } \varepsilon \to 0$$

where $\mathcal{X}_{\omega_\varepsilon}$ is the characteristic function of the set ω_ε. In particular, for any smooth function φ, defined in $\bar{\Omega}$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} f_\varepsilon \varphi = \int_\Gamma f_0\varphi.$$

Moreover, if $u^\varepsilon \to u^0$ weakly in $H^{s,p}(\Omega)$ or strongly in case of equal sign in (3.3), then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} f_\varepsilon u^\varepsilon = \int_\Gamma f_0 u^0.$$

In particular, assume $\varphi \in H^{\sigma,\rho}(\Omega)$ with $\frac{1}{\rho} < \sigma$, and denote φ_0 the trace of φ on Γ. Then

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi \to \varphi_0 \quad \text{in } H^{-s,p'}(\Omega) \text{ as } \varepsilon \to 0 \tag{3.4}$$

for any s, p such that $\frac{1}{p} < s$ and

$$(s - \frac{N}{p})_- + (\sigma - \frac{N}{\rho})_- > -N + 1, \tag{3.5}$$

where x_- denotes the negative part of x. Finally if $\varphi \in C(\overline{\Omega})$, (3.4) holds for any $s - \frac{N}{p} > -N + 1$. $\square$

Also the following particular case will be used further below.

Corollary 3.2 *Assume*

$$\|u_0^\varepsilon\|^2_{H^1(\Omega)} \leq C.$$

Then, by taking subsequences if necessary, there exists $u_0 \in H^1(\Omega)$ such that, as $\varepsilon \to 0$,

$$u_0^\varepsilon \to u_0 \quad \text{weakly in } H^1(\Omega), \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to u_{0|\Gamma} \quad \text{weakly in } H^{-1}(\Omega)$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 = \int_\Gamma |u_0|^2.$$

Proof From part A) in Lemma 3.1 we have

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \leq C\|u_0^\varepsilon\|^2_{H^1(\Omega)} \leq C.$$

Hence there exists $u_0 \in H^1(\Omega)$ such that, as $\varepsilon \to 0$, $u_0^\varepsilon \to u_0$ weakly in $H^1(\Omega)$ and by part B) in Lemma 3.1, there exists $v_0 \in L^2(\Gamma)$ such that $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to v_0$ in $H^{-1}(\Omega)$.

Since (3.3) is satisfied with $s = 1$ $p = r = 2$, again part B) in Lemma 3.1 implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 = \int_\Gamma u_0 v_0.$$

Therefore it remains to prove that $v_0 = u_{0|\Gamma}$. For this note that if $\varphi \in H^1(\Omega)$ we have, by (3.4),

$$\varphi_\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi \to \varphi_{|\Gamma} \quad \text{ in } H^{-1}(\Omega).$$

Then

$$\left\langle u_0^\varepsilon, \varphi_\varepsilon \right\rangle = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} u_0^\varepsilon \varphi$$

and the left hand side converges to

$$\left\langle u_0, \varphi_0 \right\rangle = \int_\Gamma u_0 \varphi$$

while the right hand side converges to

$$\left\langle v_0, \varphi \right\rangle = \int_\Gamma v_0 \varphi$$

Hence, $v_0 = u_{0|\Gamma}$ as claimed.$\square$

Lemma 3.1 can now be extended to handle concentrating integrals including a time dependence.

Lemma 3.3 *A) Consider $v \in L^r((0,T), H^{s,p}(\Omega))$ with $1 \leq r < \infty$, $\frac{1}{p} < s$ and $H^{s,p}(\Omega) \subset L^q(\Gamma)$, that is, $s - \frac{N}{p} \geq -\frac{(N-1)}{q}$.*

Then,

$$\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |v|^q\Big)^{r/q} \leq C\int_0^T \|v(t,\cdot)\|^r_{H^{s,p}(\Omega)}dt = \|v\|^r_{L^r((0,T),H^{s,p}(\Omega))} \tag{3.6}$$

and

$$\lim_{\varepsilon\to 0}\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |v|^q\Big)^{r/q} = \int_0^T \Big(\int_\Gamma |v|^q\Big)^{r/q} = \|v\|^r_{L^r((0,T),L^q(\Gamma))}. \tag{3.7}$$

B) Consider a family g_ε defined on $(0,T)\times\omega_\varepsilon$, such that for some $1 < q < \infty$, $1 \leq r < \infty$ and a positive constant C independent of ε,

$$\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon(t,x)|^r dx\Big)^{\frac{q}{r}} dt \leq C \tag{3.8}$$

or $\int_0^T sup_{x\in\omega_\varepsilon}|g_\varepsilon(t,x)|^q\,dt \le C$ *for the case* $r=\infty$.

Then, for every s,p *satisfying (3.3), and for every sequence converging to zero (that we still denote* $\varepsilon\to 0$*) there exists a subsequence (that we still denote the same) and a function* $g\in L^q((0,T),L^r(\Gamma))$ *(or a bounded Radon measure on* $\Gamma, g\in L^q((0,T),\mathcal{M}(\Gamma))$ *if* $r=1$*) such that*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}g_\varepsilon \to g \quad \text{in } L^q((0,T),H^{-s,p'}(\Omega)),\ \text{weakly as } \varepsilon\to 0, \tag{3.9}$$

where $\mathcal{X}_{\omega_\varepsilon}$ *is the characteristic function of the set* ω_ε. *In particular, for any smooth function* φ, *defined in* $[0,T]\times\bar{\Omega}$, *we have*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} g_\varepsilon\varphi = \int_0^T\int_\Gamma g\varphi. \tag{3.10}$$

Also, if $u^\varepsilon\to u^0$ *strongly in* $L^{q'}((0,T),H^{s,p}(\Omega))$ *then*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon = \int_0^T\int_\Gamma g u^0. \tag{3.11}$$

C) Consider a family g_ε *defined on* $(0,T)\times\omega_\varepsilon$, *and assume that for some* $1<r,q<\infty$, *there exist* $h\in L^q(0,T)$, *and* $g\in L^q((0,T),L^r(\Gamma))$ *such that*

$$\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g_\varepsilon(t,\cdot)|^r\Big)^{\frac{1}{r}} \le h(t), \quad a.e. \quad t\in[0,T] \tag{3.12}$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}g_\varepsilon(t,\cdot)\to g(t,\cdot) \text{ in } H^{-s,p'}(\Omega) \quad a.e. \quad t\in(0,T) \tag{3.13}$$

with s,p *satisfying (3.3). Then*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}g_\varepsilon \to g \quad \text{in } L^q((0,T),H^{-s,p'}(\Omega)). \tag{3.14}$$

In particular, if $\varphi\in L^q((0,T),H^{\sigma,\rho}(\Omega))$, *with* $\sigma>\frac{1}{\rho}$, *we consider* $\varphi_\varepsilon(t)=\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi(t)$ *and* $\varphi_0(t)=\varphi|_\Gamma(t)$. *Then*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi\to\varphi_0 \quad \text{in } L^q((0,T),H^{-s,p'}(\Omega)) \text{ as } \varepsilon\to 0 \tag{3.15}$$

for σ,ρ,s,p *as in (3.5). If* $\varphi\in C([0,T]\times\bar{\Omega})$, *(3.15) holds for any* $q>1$ *and* $s>\frac{1}{p}$ *with* $s-\frac{N}{p}>-N+1$.

Proof A) Observe that (3.1) gives (3.6) right away. Now, we note that for fixed $t\in[0,T]$, from (3.2) we get

$$\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|v(t,\cdot)|^q\Big)^{r/q}\le C\|v(t,\cdot)\|^r_{H^{s,p}(\Omega)} \text{ and } \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|v(t,\cdot)|^q=\int_\Gamma|v(t,\cdot)|^q.$$

Then, applying Lebesgue's dominated convergence theorem, we obtain (3.7).

B) Define, for s, p satisfying (3.3), the linear forms

$$L_\varepsilon(\varphi) = \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} g_\varepsilon \varphi$$

on $L^{q'}((0,T), H^{s,p}(\Omega))$. By Hölder's inequality we get

$$\Big|L_\varepsilon(\varphi)\Big| \le \int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon|^r\Big)^{\frac{1}{r}} \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^{r'}\Big)^{\frac{1}{r'}} \le$$

$$\le \Big[\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon|^r\Big)^{\frac{q}{r}}\Big]^{\frac{1}{q}} \Big[\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^{r'}\Big)^{\frac{q'}{r'}}\Big]^{\frac{1}{q'}}.$$

Hence using (3.8) and (3.6), we get

$$\Big|L_\varepsilon(\varphi)\Big| \le C\Big[\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^{r'}\Big)^{\frac{q'}{r'}}\Big]^{\frac{1}{q'}} \le C\|\varphi\|_{L^{q'}((0,T),H^{s,p}(\Omega))}. \tag{3.16}$$

Hence L_ε, is a bounded family in the dual space of $L^{q'}((0,T), H^{s,p}(\Omega))$. Therefore, by the Banach-Alaouglu-Bourbaki theorem, and taking subsequences if necessary, we have that there exists $L_0 \in \Big[L^{q'}((0,T), H^{s,p}(\Omega))\Big]' := L^q((0,T), H^{-s,p'}(\Omega))$ such that

$$L_\varepsilon(\varphi) \to L_0(\varphi), \qquad \text{for all } \varphi \in L^{q'}((0,T), H^{s,p}(\Omega))$$

as $\varepsilon \to 0$ and the limit is uniform for φ in compact sets of $L^{q'}((0,T), H^{s,p}(\Omega))$.

In particular, from the first inequality in (3.16) and (3.7), we get

$$|L_0(\varphi)| \le C\|\varphi\|_{L^{q'}((0,T),L^{r'}(\Gamma)))} \qquad \text{for every } \varphi \in L^{q'}((0,T), H^{s,p}(\Omega)).$$

Now taking into account that if $X \subset Y$ is dense, then $L^{q'}((0,T), X)$ is dense in $L^{q'}((0,T), Y)$ and since traces of $H^{s,p}(\Omega)$ are dense in $L^{r'}(\Gamma)$, we get

$$L^{q'}((0,T), H^{s,p}(\Omega)) \text{ is dense in } L^{q'}((0,T), L^{r'}(\Gamma)).$$

Thus, $L_0 \in (L^{q'}((0,T), L^{r'}(\Gamma)))'$ and there exist $g \in L^q((0,T), L^r(\Gamma))$ such that $L_0 = g$, i.e.

$$L_0(\varphi) = \int_0^T \int_\Gamma g\varphi$$

which proves (3.9), (3.10) and (3.11).

C) First, we note that from (3.12) together with (3.1) we have that

$$\Big|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon(t,\cdot)\varphi\Big| \le \Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon(t,\cdot)|^r\Big]^{\frac{1}{r}} \Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^{r'}\Big]^{\frac{1}{r'}} \le Ch(t)\|\varphi\|_{H^{s,p}(\Omega)},$$

that is

$$\|\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon(t,\cdot)\|_{H^{-s,p'}(\Omega)} \le Ch(t).$$

Next, taking into account (3.13) we can apply Lebesgue's dominated convergence theorem to get (3.14).

In particular, if $\varphi \in L^q((0,T), H^{\sigma,\rho}(\Omega))$, with $\sigma > \frac{1}{\rho}$, we consider $g_\varepsilon(t) = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi(t)$ and $\varphi_0(t) = \varphi|_\Gamma(t)$. Then, by (3.1), we have for a.e. $t \in (0,T)$

$$\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi(t,\cdot)|^r\Big)^{1/r} \le C\|\varphi(t,\cdot)\|_{H^{\sigma,\rho}(\Omega)} = h(t) \in L^q(0,T)$$

and by (3.4),

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi(t) \to \varphi_0(t) \quad \text{ in } H^{-s,p'}(\Omega) \text{ as } \varepsilon \to 0$$

for σ, ρ, s, p as in (3.5). Then (3.12) and (3.13) are satisfied.

If $\varphi \in C([0,T]\times\bar{\Omega})$, denote $h(t) = sup_{x\in\bar{\Omega}}|\varphi(t,x)|$. Then for any $1 \le r, q < \infty$, taking into account that $|\omega_\varepsilon| \le C\varepsilon$ for some $C > 0$, we obtain

$$\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi(t,x)|^r dx\Big)^{\frac{1}{r}} \le Ch(t) \in L^q(0,T).$$

Also, for fixed $t \in [0,T]$, by (3.4) we have

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi(t,\cdot) \to \varphi_0(t,\cdot), \quad \text{ as } \varepsilon \to 0, \text{ in } H^{-s,p'}(\Omega).$$

for any s, p with $s > \frac{1}{p}$ and $s - \frac{N}{p} > -N+1$. Then, we can choose $r > 1$ such that $s - \frac{N}{p} > -\frac{N-1}{r'}$ and then (3.12) and (3.13) are satisfied again. □

Remark 3.4 *The results in parts, B) and C) of Lemma 3.3 also hold with minor changes when either $r = 1$ or $q = 1$. Since in the proof above $L^{q'}$ and $L^{r'}$ appear, in such a case some spaces of measures enter in the result. Also, when, $\varphi \in C([0,T]\times\bar{\Omega})$ it can be actually shown that (3.15) holds for $r = \infty$.*

For the sake of simplicity in the exposition we have not included these cases.

Now we prove the following result that will be used below in the analysis of (1.1) and (1.2). Note that the assumption on the potentials below is, not only uniform in ε, but more restrictive in ρ than the one needed for fixed ε, as in (2.2).

Lemma 3.5 *Assume that the potentials V_ε satisfy*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho \le C, \text{ with } \rho > N-1$$

and assume, that after taking some subsequence, if necessary, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon\varphi = \int_\Gamma V\varphi$$

for any smooth function φ defined in $\bar{\Omega}$ and for some function $V \in L^\rho(\Gamma)$; see Lemma 3.1, part B). Then

i) There exists some $\lambda_0 > 0$, independent of $\varepsilon > 0$, such that for $\lambda > \lambda_0 > 0$ the elliptic operator, associated to the parabolic problems (1.1) and (1.2), are positive.

ii) If s is such that $\frac{1}{2} + \frac{N-1}{2\rho} < s \leq 1$ and

$$u^\varepsilon \to u^0 \text{ weakly in } L^2((0,T), H^s(\Omega)),$$

then for any function $\varphi \in L^2((0,T), H^s(\Omega))$ we have

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \varphi \to \int_0^T \int_\Gamma V u^0 \varphi$$

Proof:

i) We will prove there exists λ_0 such that the bilinear forms in $H^1(\Omega)$

$$a_\varepsilon(\phi,\xi) = \frac{1}{2}\int_\Omega \nabla\phi\nabla\xi + \lambda\int_\Omega \phi\xi + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon \phi\xi$$

and

$$a_0(\phi,\xi) = \frac{1}{2}\int_\Omega \nabla\phi\nabla\xi + \lambda\int_\Omega \phi\xi + \int_\Gamma V\phi\xi$$

are uniformly coercive for $\lambda > \lambda_0$.

For this, note that for every $\phi \in H^1(\Omega)$ and for the negative parts $(V_\varepsilon)_-$ we have the bound

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} (V_\varepsilon)_-|\phi|^2 \leq \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |(V_\varepsilon)_-|^\rho\Big)^{\frac{1}{\rho}}\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\phi|^{2\rho'}\Big]^{\frac{1}{\rho'}} \leq C\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\phi|^{2\rho'}\Big]^{\frac{1}{\rho'}}. \tag{3.17}$$

Now, since $\rho > N-1$, there exists $\frac{N-1}{2\rho} + \frac{1}{2} \leq s < 1$ such that $H^s(\Omega) \subset L^{2\rho'}(\Gamma)$ and from Lemma 3.1 and interpolation, we have that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} (V_\varepsilon)_-|\phi|^2 \leq C\|\phi\|^2_{H^s(\Omega)} \leq C\|\phi\|^{2s}_{H^1(\Omega)}\|\phi\|^{2(1-s)}_{L^2(\Omega)}.$$

Finally using Young's inequality, we get for any $\delta > 0$

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} (V_\varepsilon)_-|\phi|^2 \leq \delta\|\phi\|^2_{H^1(\Omega)} + C_\delta\|\phi\|^2_{L^2(\Omega)}.$$

Hence, we can take δ small enough and λ large enough such that

$$a_\varepsilon(\phi,\phi) \geq C\|\phi\|^2_{H^1(\Omega)} \text{ with } C = C(\lambda) > 0 \text{ independent of } \varepsilon.$$

A similar argument using that $V \in L^\rho(\Gamma)$ and $\rho > N-1$ gives the result for the bilinear form a_0. Notice that in this case we have an estimate completely similar to (3.17), now with boundary integrals.

ii) First, for $s > \frac{1}{p}$, $\sigma > \frac{1}{q}$ and $(s - \frac{N}{p})_- + (\sigma - \frac{N}{q})_- > -\frac{N-1}{\rho'}$, we define the operators, $P_\varepsilon : H^{s,p}(\Omega) \mapsto, H^{-\sigma,q}(\Omega)$ for $0 \le \varepsilon \le \varepsilon_0$ by

$$< P_\varepsilon(u), \varphi >= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon u\varphi, \qquad < P_0(u), \varphi >= \int_\Gamma V u\varphi.$$

Then from Lemma 2.5 in [2] we get $P_\varepsilon \to P_0$ in $\mathcal{L}(X, Y)$ with $X = H^{s,p}(\Omega)$ and $Y = H^{-\sigma,q}(\Omega))$.

Now we consider $\sigma = s$, $p = q = 2$ and so $X = H^s(\Omega)$ and $Y = H^{-s}(\Omega)$. This choice is possible provided $2(s - \frac{N}{2})_- > -\frac{N-1}{\rho'}$, which leads to the lower bound on s in the statement. Note that this lower bound is compatible with $s \le 1$ because $\rho > N - 1$.

Then, by Lemma 3.6 below, we have that $P_\varepsilon u^\varepsilon \to P_0 u^0$ weakly in $L^2((0,T), Y)$. In particular for any function $\varphi \in L^2((0,T), Y') = L^2((0,T), H^s(\Omega))$ we have

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon\varphi \to \int_0^T\int_\Gamma V u^0\varphi$$

and we conclude.$\square$

Now we prove the result used above.

Lemma 3.6 *Assume X and Y are reflexive Banach spaces and $P_\varepsilon \to P_0$ in $\mathcal{L}(X, Y)$.*

Then, if $u^\varepsilon \to u^0$ weakly in $L^2((0,T), X)$, then

$$P_\varepsilon u^\varepsilon \to P_0 u^0 \text{ weakly in } L^2((0,T), Y).$$

Proof First note that if $\int_0^T \|u^\varepsilon(t)\|_X^2 = \|u^\varepsilon\|^2_{L^2((0,T),X)} \le C$ then

$$\|P_\varepsilon u^\varepsilon - P_0 u^\varepsilon\|^2_{L^2((0,T),Y)} \le \int_0^T \|P_\varepsilon - P_0\|^2_{\mathcal{L}(X,Y)}\|u^\varepsilon(t)\|_X^2 dt \le C\|P_\varepsilon - P_0\|^2_{\mathcal{L}(X,Y)} \to 0 \text{ as } \varepsilon \to 0.$$

Now assume $u^\varepsilon \to u^0$ weakly in $L^2((0,T), X)$, and take $\phi \in L^2((0,T), Y')$, then

$$\Big|\int_0^T < P_\varepsilon u^\varepsilon, \phi >_{Y,Y'} - < P_0 u^0, \phi >_{Y,Y'}\Big| \le$$

$$\le \Big|\int_0^T < P_\varepsilon u^\varepsilon, \phi >_{Y,Y'} \pm < P_0 u^\varepsilon, \phi >_{Y,Y'} - < P_0 u^0, \phi >_{Y,Y'}\Big| \le (1) + (2)$$

where

$$(1) \equiv \Big|\int_0^T < P_\varepsilon u^\varepsilon, \phi >_{Y,Y'} - P_0 u^\varepsilon, \phi >_{Y,Y'}\Big|$$

and

$$(2) \equiv \Big|\int_0^T < P_0 u^\varepsilon, \phi >_{Y,Y'} - < P_0 u^0, \phi >_{Y,Y'}\Big|.$$

Thus, we obtain

$$(1) \le \Big|\int_0^T < P_\varepsilon u^\varepsilon - P_0 u^\varepsilon, \phi >_{Y,Y'}\Big| \le \int_0^T \|P_\varepsilon u^\varepsilon - P_0 u^\varepsilon\|_Y \|\phi\|_{Y'} dt$$

and we get (1) $\to 0$ as $\varepsilon \to 0$. Moreover, we have that

$$(2) \le \Big| \int_0^T < P_0(u^\varepsilon - u^0), \phi >_{Y,Y'} \Big| = \Big| \int_0^T < u^\varepsilon - u^0, P_0^* \phi >_{X,X'} \Big|$$

with $P_0^*\phi \in L^2((0,T), X')$. Then using $u^\varepsilon \to u^0$ weakly in $L^2((0,T), X)$ we get also (2) $\to 0$ as $\varepsilon \to 0$.$\square$

We also have the following result.

Lemma 3.7 *We consider a family functions $u^\varepsilon : [0,T] \to H^1(\Omega)$ such that for some positive constant C independent of ε and t, we have*

$$\|u^\varepsilon(t,\cdot)\|_{H^1(\Omega)} \le C, \quad t \in [0,T] \tag{3.18}$$

and $u_t^\varepsilon \in L^2((0,T) \times \omega_\varepsilon)$ with

$$\frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u_t^\varepsilon|^2 \le C. \tag{3.19}$$

Then, there exists a subsequence (that we still denote the same) and a function $u^0 \in L^\infty((0,T), H^1(\Omega))$ with $u^0_{|\Gamma} \in H^1((0,T), L^2(\Gamma))$ such that as $\varepsilon \to 0$,

$$u^\varepsilon \to u^0 \quad w - * \quad in\ L^\infty((0,T), H^1(\Omega))$$

and

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma} \qquad in\ H^1((0,T), H^{-1}(\Omega)).$$

In particular, for every $\varphi \in L^2((0,T), H^1(\Omega))$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} u^\varepsilon \varphi = \int_0^T \int_\Gamma u^0 \varphi, \tag{3.20}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} u_t^\varepsilon \varphi = \int_0^T \int_\Gamma u_t^0 \varphi, \tag{3.21}$$

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma} \qquad in\ C([0,T], H^{-1}(\Omega))\ if\ \varepsilon \to 0 \tag{3.22}$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 = \int_0^T \int_\Gamma |u^0|^2.$$

Proof: We prove this result in several steps.
Step 1. First, since $u^\varepsilon \in L^\infty((0,T), H^1(\Omega))$ is bounded, by taking subsequences if necessary, we can assume that it converges *weak** in $L^\infty((0,T), H^1(\Omega))$ to u^0; that is

$$\Big\langle u^\varepsilon, \varphi \Big\rangle \to \Big\langle u^0, \varphi \Big\rangle \quad \text{ if } \varepsilon \to 0 \quad \forall \varphi \in L^1((0,T), H^{-1}(\Omega)).$$

Step 2. From (3.18) and (3.6), with $s = 1$, $p = 2$, $q = r = 2$, we have

$$\frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 \le C \int_0^T \|u^\varepsilon\|^2_{H^1(\Omega)} \le C.$$

This and (3.19) implies, using Part B) in Lemma 3.3 (with $q = 2 = r$), that $W^\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon$ is uniformly bounded in $H^1((0,T), H^{-1}(\Omega)) \subset C([0,T], H^{-1}(\Omega))$.

Therefore, by taking subsequences again, if necessary, we can assume that

$$W^\varepsilon \to W^0 \quad \text{weakly in } H^1((0,T), H^{-1}(\Omega)).$$

At the same time from Part B) in Lemma 3.3 (with $q = 2 = r$), we get that

$$W^0 \in H^1((0,T), L^2(\Gamma)).$$

Step 3. We will prove now, $W^0 = u^0_{|\Gamma}$ and then we get (3.20) and (3.21).

For this, consider $\varphi \in L^2((0,T), H^1(\Omega))$ and then (3.15) gives

$$\varphi_\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi \to \varphi_0 = \varphi_{|\Gamma} \quad \text{in } L^1((0,T), H^{-1}(\Omega)) \quad \text{as } \varepsilon \to 0$$

and then from Step 1

$$\Big\langle u^\varepsilon, \varphi_\varepsilon \Big\rangle = \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} u^\varepsilon \varphi = \Big\langle W^\varepsilon, \varphi \Big\rangle.$$

Then the left hand side converges to

$$\Big\langle u^0, \varphi_0 \Big\rangle = \int_0^T \int_\Gamma u^0 \varphi$$

while the right hand side converges to

$$\Big\langle W^0, \varphi \Big\rangle.$$

Hence, $W^0 = u^0_{|\Gamma}$ as claimed.

Step 4. Now we prove (3.22) and for this we use Ascoli-Arzela's Theorem. First, we note that W^ε_t is uniformly bounded in $L^2((0,T), H^{-1}(\Omega))$ and then $W^\varepsilon(t,\cdot)$ is equicontinuous in $H^{-1}(\Omega), t \in (0,T)$. Second, we will prove that $W^\varepsilon(t,\cdot)$ is uniformly bounded in $X = H^{-s}(\Omega)$ for some $s < 1$. Since $X \subset H^{-1}(\Omega)$ is compact, we conclude the proof.

In effect, take $r > 2$ such that $H^1(\Omega) \subset L^r(\Gamma)$ and $s < 1$ such that $H^s(\Omega) \subset L^{r'}(\Gamma)$, i.e. $-\frac{N-1}{r'} < s - \frac{N}{2} < 1 - \frac{N}{2}$. Then by Lemma 3.1

$$\Big|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} u^\varepsilon \varphi\Big| \le \Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon|^r\Big]^{\frac{1}{r}} \Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^{r'}\Big]^{\frac{1}{r'}} \le C\|u^\varepsilon\|_{H^1(\Omega)}\|\varphi\|_{H^s(\Omega)} \le C\|\varphi\|_{H^s(\Omega)}.$$

That is, $\|W^\varepsilon(t,\cdot)\|_X \le C$ and we conclude.

The last property in the statement follows from the weak convergence of u^ε and the strong convergence of $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon$. $\square$

We will finally make use of the following

Lemma 3.8 *Assume the family of potentials V_ε is as in Lemma 3.5 but with*

$$\rho > 2(N-1). \tag{3.23}$$

Also, assume u^ε is as in Lemma 3.7, that is, satisfies (3.18) and (3.19), and let u^0 be as in the conclusion of Lemma 3.7.

Then if s is such that $\frac{1}{2} + \frac{N-1}{\rho} < s \le 1$, we have

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \to V u^0_{|\Gamma} \quad \textit{strongly in } L^2((0,T), H^{-s}(\Omega)) \tag{3.24}$$

and

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon |u^\varepsilon|^2 \to \int_0^T\int_\Gamma V|u^0|^2.$$

Proof Observe that once (3.24) is proved, we have

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon |u^\varepsilon|^2 = \Big\langle \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon, u^\varepsilon\Big\rangle \to \Big\langle V u^0_{|\Gamma}, u^0\Big\rangle = \int_0^T\int_\Gamma V|u^0|^2$$

and we conclude.

Now, to prove (3.24) we use Ascoli-Arzela's Theorem like in the Lemma 3.7. For this, denote $W^\varepsilon = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon$. Since from (3.23), $\rho > N-1$, then for any $\frac{1}{2} + \frac{N-1}{2\rho} < s \le 1$ we have $H^s(\Omega) \subset L^{2\rho'}(\Gamma)$ and

$$\Big|\Big\langle \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon, \varphi\Big\rangle\Big| \le \Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^\rho\Big]^{\frac{1}{\rho}}\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u^\varepsilon|^{2\rho'}\Big]^{\frac{1}{2\rho'}}\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^{2\rho'}\Big]^{\frac{1}{2\rho'}} \le C\|u^\varepsilon\|_{H^s(\Omega)}\|\varphi\|_{H^s(\Omega)}.$$

Therefore, from (3.18), W^ε is uniformly bounded in $L^\infty((0,T), H^{-s}(\Omega))$.

Now observe that from (3.19) we have that $W^\varepsilon_t = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon_t$ satisfies

$$\Big|\Big\langle \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon_t, \varphi\Big\rangle\Big| \le \Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^\rho\Big]^{\frac{1}{\rho}}\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u^\varepsilon_t|^2\Big]^{\frac{1}{2}}\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^r\Big]^{\frac{1}{r}} \le C\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u^\varepsilon_t|^2\Big]^{\frac{1}{2}}\Big[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^r\Big]^{\frac{1}{r}}$$

with $\frac{1}{\rho} + \frac{1}{2} + \frac{1}{r} = 1$ i.e. $r = \frac{2\rho}{\rho-2}$.

Now, from (3.23), i.e. $\rho > 2(N-1)$ for any s such that $\frac{1}{2} + \frac{N-1}{\rho} < s \le 1$ we have that $H^s(\Omega) \subset L^{\frac{2\rho}{\rho-2}}(\Gamma)$ and then have that, integrating in time in the inequality above

$$\Big|\int_0^T \Big\langle \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon_t, \varphi\Big\rangle\Big| \le C\Big[\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}|u^\varepsilon_t|^2\Big]^{\frac{1}{2}}\Big[\int_0^T\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^r\Big)^{\frac{2}{r}}\Big]^{\frac{1}{2}} \le C\Big[\int_0^T\|\varphi\|^2_{H^s(\Omega)}\Big]^{\frac{1}{2}}$$

where we have used (3.19) and (3.1) in Lemma 3.1.

Hence, $W^\varepsilon_t = \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon_t$ is uniformly bounded in $L^2((0,T), H^{-s}(\Omega))$. In particular,$W^\varepsilon$ is uniformly bounded in $H^1((0,T), H^{-s}(\Omega)) \subset C([0,T], H^{-s}(\Omega))$ and W^ε is equicontinuous with values in $Y = H^{-s}(\Omega)$.

Now we will prove that $W^\varepsilon(t,\cdot)$ is uniformly bounded in $\tilde{Y} = H^{-s^*}(\Omega)$ for some $s^* < s$. Since $\tilde{Y} = H^{-s^*}(\Omega) \subset Y = H^{-s}(\Omega)$ is compact, we conclude the proof.

In effect,we note that if s satisfies $\frac{1}{2} + \frac{N-1}{\rho} < s \le 1$, then there exists s^* satisfying $\frac{1}{2} + \frac{N-1}{2\rho} < s^* < \frac{1}{2} + \frac{N-1}{\rho} < s \le 1$, i.e. $H^{s^*}(\Omega) \subset L^{\frac{2\rho}{\rho-1}}(\Gamma) = L^{2\rho'}(\Gamma)$ and by the first part of the proof we conclude. $\square$.

4 Singular limit as $\varepsilon \to 0$

We analyze the limit of the solutions of the parabolic problems (1.1), with $0 \leq \varepsilon \leq \varepsilon_0$. For this we will assume that the data of the problem satisfy

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho \leq C, \quad \rho > N-1, \tag{4.1}$$

$$u_0^\varepsilon \in H^1(\Omega) \quad \text{and } \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \leq C \tag{4.2}$$

$$f_\varepsilon \in L^2((0,T), H^{-1}(\Omega)), \text{ and } \int_0^T \|f_\varepsilon\|^2_{H^{-1}(\Omega)} \leq C \tag{4.3}$$

and for some $r \geq \max\left\{1, \frac{2(N-1)}{N}\right\}$

$$\int_0^T \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g_\varepsilon|^r\Big)^{\frac{2}{r}} \leq C \tag{4.4}$$

for some constant C independent of ε.

Then, by Lemma 3.1 and 3.3, by taking subsequences if necessary, we can assume that there exists functions $V \in L^\rho(\Gamma)$, $v_0 \in L^2(\Gamma)$, $f \in L^2((0,T), H^{-1}(\Omega))$ and $g \in L^2((0,T), L^r(\Gamma))$ such that, as $\varepsilon \to 0$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V \text{ weakly in } H^{-s,p'}(\Omega) \text{ with } s - \frac{N}{p} > -\frac{N-1}{\rho'}, \tag{4.5}$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to v_0 \text{ weakly in } H^{-s,p'}(\Omega) \text{ with } s - \frac{N}{p} > -\frac{N-1}{2} \tag{4.6}$$

$$f_\varepsilon \to f \text{ weakly in } L^2((0,T), H^{-1}(\Omega)) \tag{4.7}$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \text{ weakly in } L^2((0,T), H^{-s,p'}(\Omega)) \text{ with } s - \frac{N}{p} > -\frac{N-1}{r'}. \tag{4.8}$$

In particular, we have that

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_0^\varepsilon \to v_0 \text{ weakly in } H^{-1}(\Omega)$$

and since $r \geq \frac{2(N-1)}{N}$, from (4.8) with $p = 2$ and $s = 1$,

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \text{ weakly in } L^2((0,T), H^{-1}(\Omega)).$$

We consider also a "generalized weak solution" solution of (1.2) defined by:

Definition 4.1 *A function $u^0 \in L^2((0,T), H^1(\Omega))$ is a "generalized weak solution" of (1.2) if it satisfies*

$$-\Delta u^0 + \lambda u^0 = f(t), \text{ a.e. } \quad t \in (0,T) \tag{4.9}$$

in the sense of distributions in Ω and for any function $\Phi \in H^1(\Omega)$

$$\frac{d}{dt}\Big(\int_\Gamma u^0\Phi\Big) + \int_\Omega \nabla u^0 \nabla\Phi + \lambda\int_\Omega u^0\Phi + \int_\Gamma V u^0\Phi = \int_\Omega f\Phi + \int_\Gamma g\Phi, \quad \text{in } \mathcal{D}'(0,T) \tag{4.10}$$

and

$$\int_\Gamma u^0(t)\Phi \to \int_\Gamma v_0\Phi \text{ as } t \to 0. \tag{4.11}$$

Then we have

Proposition 4.2 *Under the above notations, assume (4.5), (4.6), (4.7) and (4.8) and consider u^ε the solutions of (1.1) as in Definition 2.3. Moreover assume $\lambda > \lambda_0$ as in Lemma 3.5.*

Then there exists a subsequence (denoted the same) and a function $u^0 \in L^2((0,T), H^1(\Omega))$ such that as $\varepsilon \to 0$,

$$u^\varepsilon \to u^0 \text{ weakly in } L^2((0,T), H^1(\Omega))$$

and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma} \text{ in } L^2((0,T), H^{-1}(\Omega)) \text{ weakly },$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u^\varepsilon \to V u^0_{|\Gamma} \text{ in } L^2((0,T), H^{-1}(\Omega)) \text{ weakly.}$$

In particular, for any $\varphi \in L^2((0,T), H^1(\Omega))$

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} u^\varepsilon\varphi \to \int_0^T\int_\Gamma u^0\varphi,$$

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon\varphi \to \int_0^T\int_\Gamma V u^0\varphi.$$

Moreover, u^0 is a "generalized weak solution" of (1.2), in the sense of Definition 4.1 with initial data $v_0 \in L^2(\Gamma)$, potential $V \in L^\rho(\Gamma)$ and nonhomogeneous terms $f \in L^2((0,T), H^{-1}(\Omega))$ and $g \in L^2((0,T), L^r(\Gamma))$.

Proof We proceed in several steps.
Step 1. Uniform bounds for u^ε.

By considering first smooth data, multiplying the equation by u^ε in $L^2(\Omega)$, and then by a density argument, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u^\varepsilon|^2\Big) + \int_\Omega |\nabla u^\varepsilon|^2 + \lambda\int_\Omega |u^\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon|u^\varepsilon|^2 = \int_\Omega f_\varepsilon u^\varepsilon + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon. \tag{4.12}$$

Now from Lemma 3.5

$$C\|u^\varepsilon(t)\|^2_{H^1(\Omega)} \le \int_\Omega |\nabla u^\varepsilon|^2 + \lambda\int_\Omega |u^\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon|u^\varepsilon|^2$$

for some $C > 0$ independent of ε.

Next, applying Young's inequality, we obtain, for any $\delta > 0$,

$$|\int_\Omega f_\varepsilon u^\varepsilon| \leq \|u^\varepsilon\|_{H^1(\Omega)}\|f_\varepsilon\|_{H^{-1}(\Omega)} \leq \delta\|u^\varepsilon\|^2_{H^1(\Omega)} + \frac{1}{4\delta}\|f_\varepsilon\|^2_{H^{-1}(\Omega)}$$

$$|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon| \leq \Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g_\varepsilon|^r\Big)^{\frac{1}{r}}\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u^\varepsilon|^{r'}\Big)^{\frac{1}{r'}} \leq C\|u^\varepsilon\|_{H^1(\Omega)}\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g_\varepsilon|^r\Big)^{\frac{1}{r}} \leq$$

$$\leq \delta\|u^\varepsilon\|^2_{H^1(\Omega)} + \frac{C}{4\delta}(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g_\varepsilon|^r)^{\frac{2}{r}}.$$

Now, taking δ enough small and integrating (4.12) in $t \in (0, T)$ and using (4.2), (4.3), (4.4), we obtain that for $0 \leq t \leq T$,

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u^\varepsilon(t)|^2 + C\int_0^t \|u^\varepsilon(s)\|^2_{H^1(\Omega)}ds \leq \frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u_0^\varepsilon|^2 + \frac{1}{2\delta}\int_0^t \|f_\varepsilon\|^2_{H^{-1}(\Omega)} +$$

$$+\frac{C}{2\delta}\int_0^t \frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g_\varepsilon|^{\frac{2}{r}}dt \leq C.$$

Then, we have that

$$\int_0^T \|u^\varepsilon(t)\|^2_{H^1(\Omega)}dt \leq C \quad \text{and} \quad \sup_{t\in[0,T]}\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u^\varepsilon(t)|^2 \leq C. \tag{4.13}$$

Step 2. Passing to the limit.

From (4.13) and Lemma 3.3 part B) with $q = r = 2$, by taking subsequences if necessary, there exists a subsequence which converges weakly to u^0 in $L^2((0, T), H^1(\Omega))$ and there exists $w \in L^2((0, T), L^2(\Gamma)) = L^2((0, T) \times \Gamma)$ such that

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}u^\varepsilon \to w \quad \text{in } L^2((0, T), H^{-1}(\Omega)) \text{ weakly as } \varepsilon \to 0. \tag{4.14}$$

Now, we prove that $w = u^0_{|\Gamma}$. For this, note that for every $\varphi \in L^2((0, T), H^1(\Omega))$

$$\Big\langle\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}u^\varepsilon, \varphi\Big\rangle = \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}u^\varepsilon\varphi = \Big\langle u^\varepsilon, \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi\Big\rangle,$$

Then, using (3.15) and by taking another subsequence if is necessary, we obtain that

$$\Big\langle w, \varphi\Big\rangle = \lim_{\varepsilon\to 0}\Big\langle\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}u^\varepsilon, \varphi\Big\rangle = \lim_{\varepsilon\to 0}\Big\langle u^\varepsilon, \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}\varphi\Big\rangle = \Big\langle u^0, \varphi_{|\Gamma}\Big\rangle = \int_0^T\int_\Gamma u^0\varphi.$$

Thus $w = u^0_{|\Gamma}$.

Step 3. Next, we prove that u^0 satisfies the problem with dynamic boundary conditions (1.2) in the sense of Definition 4.1.

In effect, multiplying the equation from (1.1) by any smooth function $\varphi(t, x)$ we obtain

$$\Big\langle\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}u_t^\varepsilon, \varphi\Big\rangle + \int_0^T\int_\Omega \nabla u^\varepsilon\nabla\varphi + \lambda\int_0^T\int_\Omega u^\varepsilon\varphi + \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}V_\varepsilon u^\varepsilon\varphi = \int_0^T\int_\Omega f_\varepsilon\varphi + \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}g_\varepsilon\varphi$$

Now, assume $\varphi(T)=0$, using Fubbini Theorem and integrating by parts, we manipulate the term, $\left\langle \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u_t^\varepsilon, \varphi\right\rangle$ to get

$$-\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} u^\varepsilon \varphi_t - \frac{1}{\varepsilon}\int_{\omega_\varepsilon} u^\varepsilon(0,\cdot)\varphi(0,\cdot) + \int_0^T\int_\Omega \nabla u^\varepsilon \nabla\varphi + \lambda\int_0^T\int_\Omega u^\varepsilon\varphi +$$

$$+\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon\varphi = \int_0^T\int_\Omega f_\varepsilon\varphi + \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} g_\varepsilon\varphi. \tag{4.15}$$

Next, using (4.14) where $w = u^0_{|\Gamma}$ and applying (3.10) from Lemma 3.3 part B) with $q=r=2$, we have, as $\varepsilon\to 0$,

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} u^\varepsilon\varphi_t \to \int_0^T\int_\Gamma u^0\varphi, \qquad \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} g_\varepsilon\varphi \to \int_0^T\int_\Gamma g\varphi$$

and

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon u^\varepsilon\varphi \to \int_0^T\int_\Gamma V u^0\varphi.$$

Thus, taking limit as ε goes to zero in (4.15), we get

$$-\int_\Gamma v_0\varphi(0,\cdot) - \int_0^T\int_\Gamma u^0\varphi_t + \int_0^T\Big[\int_\Omega \nabla u^0\nabla\varphi + \lambda\int_\Omega u^0\varphi\Big]$$

$$+\int_0^T\int_\Gamma V u^0\varphi = \int_0^T\int_\Omega f\varphi + \int_0^T\int_\Gamma g\varphi. \tag{4.16}$$

Next, we prove that u^0 is a "generalized weak solution" of (1.1) in the sense of Definition 4.1.

I) First, we consider $\varphi(t,x)=\psi(t)\phi(x)$ with $\psi\in\mathcal{D}(0,T)$ such that $\psi(T)=\psi(0)=0$, and $\phi\in H_0^1(\Omega)$. Thus, from (4.16) we get

$$\int_0^T \psi(t)\Big[\int_\Omega \nabla u^0\nabla\phi + \lambda\int_\Omega u^0\phi - \int_\Omega f\phi\Big] = 0.$$

Since this is for all such $\psi(t)$, we get, for all $\phi\in H_0^1(\Omega)$,

$$\int_\Omega \nabla u^0\nabla\phi + \lambda\int_\Omega u^0\phi = \int_\Omega f\phi, \qquad \text{a.e. } t\in[0,T].$$

Therefore the limit function satisfies (4.9).

II) We consider now $\varphi(t,x)=\psi(t)\phi(x)$ with $\psi\in\mathcal{D}(0,T)$ such that $\psi(T)=\psi(0)=0$, and $\phi\in H^1(\Omega)$. From (4.16) we obtain that

$$-\int_0^T\psi'(t)\int_\Gamma u^0(t)\phi + \int_0^T\psi(t)\Big(\int_\Omega \nabla u^0\nabla\phi + \lambda u^0(t)\phi\Big)$$

$$+\int_0^T\psi(t)\Big(\int_\Gamma V u^0\phi\Big) = \int_0^T\psi(t)\int_\Omega f\phi + \int_0^T\psi(t)\int_\Gamma g\phi.$$

Now, taking into account that

$$\Big\langle \frac{d}{dt}\Big(\int_\Gamma u^0(t,\cdot)\phi\Big), \psi(t)\Big\rangle = -\int_0^T \psi'(t)\int_\Gamma u^0(t,\cdot)\phi$$

we get (4.10), in the sense of distributions.

Finally we prove (4.11). From (4.10) and $u^0 \in L^2((0,T),H^1(\Omega))$ we get $\frac{d}{dt}\Big(\int_\Gamma u^0\phi\Big) \in L^2(0,T)$ and $\int_\Gamma u^0\phi \in H^1(0,T) \subset C[0,T]$ for any $\phi \in H^1(\Omega)$.

Using again (4.16) with $\varphi(t,x) = \psi(t)\phi(x)$ and $\psi(T) = 0$ and denoting $A = \lim_{t\to 0}\int_\Gamma u^0(t)\phi$, we obtain that

$$-\psi(0)\int_\Gamma v_0\phi - \int_0^T \psi'(t)\int_\Gamma u^0\phi + \int_0^T \psi(t)\Big(\int_\Omega \nabla u^0\nabla\phi + \lambda\int_\Omega u^0\phi\Big)$$

$$\int_0^T \psi(t)\int_\Gamma V u^0\phi = \int_0^T \psi(t)\int_\Omega f\phi + \int_0^T \psi(t)\int_\Gamma g\phi. \tag{4.17}$$

On the other hand, integrating by parts we get

$$-\int_0^T \psi'(t)\int_\Gamma u^0\phi = \int_0^T \psi(t)\frac{d}{dt}\Big(\int_\Gamma u^0\phi\Big) + \psi(0)A.$$

Using this and (4.10) in (4.17) we have that

$$A = \lim_{t\to 0}\int_\Gamma u^0(t)\phi = \int_\Gamma v_0\phi$$

and we conclude.$\square$

Remark 4.3 *Observe that as we assume (4.5), (4.6), (4.7) and (4.8) in Proposition 4.2, from any subsequence in u^ε there is another subsequence that converges to some u^0, which is a generalized weak solution of (1.2), in the sense of Definition 4.1 with the same data $v_0 \in L^2(\Gamma)$, $V \in L^\rho(\Gamma)$, $f \in L^2((0,T),H^{-1}(\Omega))$ and $g \in L^2((0,T),L^r(\Gamma))$.*

In particular, if there is just one such solution, all the family u^ε would converge to u^0.

This uniqueness result would be guaranteed below under stronger assumptions on the data.

In fact, we have

Lemma 4.4 *Assume u^0 is a "generalized weak solution" of (1.2), in the sense of Definition 4.1 with initial data $v_0 \in L^2(\Gamma)$, potential $V \in L^\rho(\Gamma)$ and nonhomogeneous terms $f \in L^1((0,T),L^2(\Omega))$ and $g \in L^1((0,T),H^{-1/2}(\Gamma))$. Assume moreover that*

$$\gamma(u^0) \in C([0,T],H^{-1/2}(\Gamma)).$$

Then u^0 is the unique solution of (1.2) in the sense of Definition 2.8.

Proof Recall that if $h = f_\Omega + g_\Gamma \in H^{-1}(\Omega)$ and $f \in L^2(\Omega)$, then $h_1 = g - \frac{\partial D_0(f)}{\partial n} \in H^{-\frac{1}{2}}(\Gamma)$ and $h_2 = f \in H_0^{-1}(\Omega)$; see right above Definition 2.8.

Since u^0 is a "generalized weak solution" of (1.2), in the sense of Definition 4.1 then (4.9) holds which implies (2.14).

Using this and taking in (4.10) a test function $\Phi \in H^1(\Omega)$, and using the results in Proposition 2.7, we get that $v(t) = \gamma(u^0(t)) \in H^{\frac{1}{2}}(\Gamma)$ satisfies

$$\frac{d}{dt}\Big(\int_\Gamma v(t)\Phi\Big) + < v(t), A_0\Phi > = < g - \frac{\partial D_0(f(t))}{\partial n}, \Phi > = < h_1(t), \Phi > .$$

Now, according to [3], taking $X = H^{-1/2}(\Gamma)$, as soon as $h_1 \in L^1((0,T), X)$, if $v \in C([0,T], X)$, $v(0) = u_0^0$ and for every $\Phi \in H^1(\Omega)$, $< v(t), \Phi >$ is absolutely continuous and

$$\frac{d}{dt} < v(t), \Phi > + < v(t), A_0\Phi > = < h_1(t), \Phi >, \quad \text{a.e. } t \in (0,T)$$

where $< \cdot, \cdot >$ denotes de pairing between X and its dual $X' = H^{1/2}(\Gamma)$, then v is given by (2.15). In such a case u^0 is the unique solution of (1.2) in the sense of Definition 2.8.

Note that by the assumptions we have both $h_1 = g - \frac{\partial D_0(f(t))}{\partial n} \in L^1((0,T), H^{-1/2}(\Gamma))$ and $v = \gamma(u^0) \in C([0,T], H^{-1/2}(\Gamma))$. $\square$

Now we impose stronger assumptions than (4.1)–(4.4) on the data and obtain stronger convergence of solutions than in Proposition 4.2. We also will obtain that all the family u^ε converges as in Remark 4.3. More precisely, we assume now the initial conditions satisfy that

$$\|u_0^\varepsilon\|^2_{H^1(\Omega)} \leq C, \tag{4.18}$$

and also the compatibility conditions on the initial data, (2.8) with $h = f_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_\varepsilon g_\varepsilon$, i.e.

$$-\Delta u_0^\varepsilon + \lambda u_0^\varepsilon = f_\varepsilon(0) \text{ in } \Omega \setminus \bar{\omega}_\varepsilon. \tag{4.19}$$

We also assume

$$f_\varepsilon \in H^1((0,T), L^2(\Omega)), \text{ and } \|f_\varepsilon\|_{H^1((0,T),L^2(\Omega))} \leq C \tag{4.20}$$

and

$$\frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon} |g_\varepsilon|^2 \leq C \tag{4.21}$$

where C is a positive constant independent of ε.

Finally, we assume (4.1) and that $\lambda > \lambda_0$, as in Lemma 3.5.

Hence using that (3.1) in Lemma 3.1 we have that $\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \leq C\|u_0^\varepsilon\|^2_{H^1(\Omega)}$ and therefore (4.18), (4.20) and (4.21) imply (4.2), (4.3) and (4.4) respectively (with $r = 2$ in the latter case).

Then by taking subsequences if necessary, we can assume (4.5), (4.6), (4.7) and (4.8). Moreover from Corollary 3.2 we have that in this case

$$u_0^\varepsilon \to u_0^0 \text{ weakly in } H^1(\Omega) \quad \text{and} \quad \frac{1}{\varepsilon}\mathcal{X}_\varepsilon u_0^\varepsilon \to u^0|_\Gamma \text{ weakly in } H^{-1}(\Omega). \tag{4.22}$$

In particular $v_0 = u^0|_\Gamma$ in (4.6).

Also, in (4.7) we have $f \in H^1((0,T), L^2(\Omega))$ and

$$f_\varepsilon \to f \text{ weakly in } H^1((0,T), L^2(\Omega)) \tag{4.23}$$

while in (4.8), with $r = r' = 2$, we have $g \in L^2((0,T), L^2(\Omega))$.

Finally, we have (4.5) with $V \in L^\rho(\Omega)$, $\rho > N-1$.

Then we first make the following remark.

Lemma 4.5 *Under the above assumptions,*

$$-\Delta u_0^0 + \lambda u_0^0 = f(0) \qquad \text{in } \Omega \tag{4.24}$$

is satisfied.

Proof We first show that

$$f_\varepsilon(0) \to f(0) \text{ in } H^{-s}(\Omega), \qquad 0 < s < 1$$

and for this we use Ascoli-Arzela's Theorem. Observe that from (4.20) we have that $(f_\varepsilon)_t$ is uniformly bounded in $L^2((0,T), H^{-s}(\Omega))$ for $0 < s < 1$ and then f_ε is equicontinuous in $H^{-s}(\Omega)$, $t \in (0,T)$. Second, from (4.20), we have that $f_\varepsilon \in H^1((0,T), L^2(\Omega)) \subset C([0,T], L^2(\Omega))$ and therefore

$$\sup_{0 \le t \le T} \|f_\varepsilon(t)\|_{L^2(\Omega)} \le C.$$

Hence $f_\varepsilon(t,\cdot)$ is uniformly bounded in $L^2(\Omega)$.

Finally, since $L^2(\Omega) \subset H^{-s}(\Omega)$ is compact, we conclude that $f_\varepsilon \to f$ in $C([0,T], H^{-s}(\Omega))$, and the convergence of $f_\varepsilon(0)$ follows.

Now to prove (4.24) we consider $\varphi \in \mathcal{D}(\Omega)$ and enough small ε such that $supp(\varphi) \subset \Omega \setminus \omega_\varepsilon$. Thus, from (4.19) we have

$$\int_\Omega \nabla u_0^\varepsilon \nabla \varphi + \lambda \int_\Omega u_0^\varepsilon \varphi = \int_\Omega f_\varepsilon(0)\varphi$$

and taking the limit $\varepsilon \to 0$, using $u_0^\varepsilon \to u_0^0$ weakly in $H^1(\Omega)$ and the convergence of $f_\varepsilon(0)$, we obtain that

$$\int_\Omega \nabla u_0^0 \nabla \varphi + \lambda \int_\Omega u_0^0 \varphi = \int_\Omega f(0)\varphi$$

and we conclude. $\square$

Hence, we have the following

Theorem 4.6 *Under the above notations, assume (4.18),(4.19),(4.20) and (4.21). Moreover assume $\lambda > \lambda_0$ as in Lemma 3.5.*

By taking subsequences if necessary, we can assume the data satisfies (4.5), (4.6), (4.7) and (4.8) and moreover (4.22), (4.23).

Finally consider u^ε the solutions of (1.1) as in Definition 2.3.

Then, u^ε (and not only a subsequence) converges as in Proposition 4.2 to a function u^0 which is the unique solution of (1.2) in the sense of Definition 2.8.

Also, u^ε converges to u^0, weak in $L^\infty((0,T),H^1(\Omega))$ and $u^0_{|\Gamma} \in H^1((0,T),L^2(\Gamma))$ and*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon \to u^0_{|\Gamma}$$

weakly in $H^1((0,T),H^{-1}(\Omega))$ and strongly in $C(([0,T],H^{-1}(\Omega))$.

If additionally $\rho > 2(N-1)$ then u^ε converges to u^0 also in $L^2((0,T),H^1(\Omega))$.

Proof We proceed in several steps.

Step 1. Uniform bounds on u^ε.

We note that we are under the assumptions of Theorem 2.4 and Proposition 2.5 and from (2.10) with $h = f_\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_\varepsilon$, we have

$$\frac{2}{\varepsilon}\int_0^t\int_{\omega_\varepsilon}|u^\varepsilon_t|^2 + \int_\Omega |\nabla u^\varepsilon|^2 + \lambda\int_\Omega |u^\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon |u^\varepsilon|^2 =$$

$$= \int_\Omega |\nabla u^\varepsilon_0|^2 + \lambda\int_\Omega |u^\varepsilon_0|^2 + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon |u^\varepsilon_0|^2 + \frac{2}{\varepsilon}\int_0^t\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon_t +$$

$$+ 2\Big(\int_0^t\int_{\omega_\varepsilon} f_\varepsilon u^\varepsilon_t + \int_{\Omega\setminus\bar\omega_\varepsilon} f_\varepsilon(t)u^\varepsilon(t) - \int_{\Omega\setminus\bar\omega_\varepsilon} f_\varepsilon(0)u^\varepsilon_0 - \int_0^t\int_{\Omega\setminus\bar\omega_\varepsilon}(f_\varepsilon)_t u^\varepsilon\Big) \qquad (4.25)$$

Now, $(f_\varepsilon)_t \in L^2((0,T),L^2(\Omega))$ and integrating by parts we obtain

$$\int_0^t\int_{\omega_\varepsilon} f_\varepsilon u^\varepsilon_t = -\int_0^t\int_{\omega_\varepsilon}(f_\varepsilon)_t u^\varepsilon + \int_{\omega_\varepsilon} f_\varepsilon(t)u^\varepsilon(t) - \int_{\omega_\varepsilon} f_\varepsilon(0)u^\varepsilon_0.$$

Hence, using Lemma 3.5, from (4.25) we have

$$\frac{2}{\varepsilon}\int_0^t\int_{\omega_\varepsilon}|u^\varepsilon_t|^2 + C\|u^\varepsilon(t)\|^2_{H^1(\Omega)} \le C\|u^\varepsilon_0\|^2_{H^1(\Omega)} + \frac{2}{\varepsilon}\int_0^t\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon_t +$$

$$+ 2\Big(\int_\Omega f_\varepsilon(t)u^\varepsilon(t) - \int_\Omega f_\varepsilon(0)u^\varepsilon_0 - \int_0^t\int_\Omega (f_\varepsilon)_t u^\varepsilon\Big). \qquad (4.26)$$

Next, applying Young's inequality we get that

$$\Big|\frac{1}{\varepsilon}\int_0^t\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon_t\Big| \le \frac{1}{\delta\varepsilon}\int_0^t\int_{\omega_\varepsilon}|g_\varepsilon|^2 + \delta\frac{1}{\varepsilon}\int_0^t\int_{\omega_\varepsilon}|u^\varepsilon_t|^2$$

for any $\delta > 0$. Using now

$$\Big|\int_\Omega f_\varepsilon(t)u^\varepsilon(t) - \int_\Omega f_\varepsilon(0)u^\varepsilon_0\Big| \le \|f_\varepsilon(t)\|_{L^2(\Omega)}\|u^\varepsilon(t)\|_{L^2(\Omega)} + \|u^\varepsilon_0\|_{L^2(\Omega)}\|f_\varepsilon(0)\|_{L^2(\Omega)},$$

and applying again the Young inequality we get

$$\Big|\int_\Omega f_\varepsilon(t)u^\varepsilon(t) - \int_\Omega f_\varepsilon(0)u^\varepsilon_0\Big| \le \delta\|u^\varepsilon(t)\|^2_{H^1(\Omega)} + \frac{1}{\delta}\|f_\varepsilon(t)\|^2_{L^2(\Omega)} + \delta\|u^\varepsilon_0\|^2_{H^1(\Omega)} + \frac{1}{\delta}\|f_\varepsilon(0)\|^2_{L^2(\Omega)}$$

and working as above

$$\Big|\int_0^t\int_\Omega (f_\varepsilon)_t u^\varepsilon\Big| \le \delta\int_0^t \|u^\varepsilon\|^2_{H^1(\Omega)} + \frac{1}{\delta}\int_0^t \|(f_\varepsilon)_t\|^2_{L^2(\Omega)}.$$

Using these inequalities, from (4.26) we have that

$$\frac{2(1-\delta)}{\varepsilon}\int_0^t\int_{\omega_\varepsilon}|u_t^\varepsilon|^2 + (C-2\delta)\|u^\varepsilon(t)\|^2_{H^1(\Omega)} \le (C+2\delta)\|u_0^\varepsilon\|^2_{H^1(\Omega)} + \frac{2}{\delta}\|f_\varepsilon(t)\|^2_{L^2(\Omega)}+$$

$$+\frac{2}{\delta}\|f_\varepsilon(0)\|^2_{L^2(\Omega)} + 2\delta\int_0^t\|u^\varepsilon\|^2_{H^1(\Omega)} + \frac{2}{\delta}\int_0^t\|(f_\varepsilon)_t\|^2_{L^2(\Omega)} + \frac{2}{\delta\varepsilon}\int_0^t\int_{\omega_\varepsilon}|g_\varepsilon|^2. \tag{4.27}$$

Now from (4.20), and denoting $y(T)=\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2_{H^1(\Omega)}$ we get

$$\delta\int_0^t\|u^\varepsilon\|^2_{H^1(\Omega)} + \frac{1}{\delta}\int_0^t\|(f_\varepsilon)_t\|^2_{L^2(\Omega)} \le T\delta y(T) + \frac{1}{\delta}C.$$

Also from (4.20), we have that $f_\varepsilon\in H^1((0,T),L^2(\Omega))\subset \mathcal{C}([0,T],L^2(\Omega))$ and therefore

$$\sup_{0\le t\le T}\|f_\varepsilon(t)\|_{L^2(\Omega)}\le C.$$

Thus, from (4.27) and using also (4.21) we obtain

$$\frac{2(1-\delta)}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}|u_t^\varepsilon|^2 + [C-2\delta(1+T)]y(T)\le C.$$

Finally, taking $\delta<\min\{1,\frac{C}{2(1+T)}\}$ we conclude that

$$\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2_{H^1(\Omega)}\le C, \quad \text{and} \quad \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}|u_t^\varepsilon|^2\le C. \tag{4.28}$$

Step 2. Passing to the limit.

First, note that we are under the assumptions of Proposition 4.2. Hence from any subsequence in u^ε there exists a subsequence (that we denote the same) that converges to some u^0 as in Proposition 4.2.

Next, from (4.28) we can apply Lemma 3.7 and then we can assume that u^ε also converges to u^0 *weak** in $L^\infty((0,T),H^1(\Omega))$, u^0 in $L^\infty((0,T),H^1(\Omega)$, $u^0_{|\Gamma}\in H^1((0,T),L^2(\Gamma))$ and

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}u^\varepsilon \to u^0_{|\Gamma} \quad \text{in } H^1((0,T),H^{-1}(\Omega))\subset C([0,T],H^{-1}(\Omega)).$$

But then such u^0 is in the situation of Lemma 4.4 and it is therefore the unique solution of (1.2) in the sense of Definition 2.8. As a consequence, all the family u^ε converges to u^0 as in above; see Remark 4.3.

Step 3. To conclude we prove the convergence in $L^2((0,T),H^1(\Omega))$ provided $\rho>2(N-1)$. For this, since we have weak convergence it is enough to prove convergence of the norm, that is, $\|u^\varepsilon\|_{L^2((0,T),H^1(\Omega))}\to\|u^0\|_{L^2((0,T),H^1(\Omega))}$ as $\varepsilon\to 0$.

Integrating in $t \in (0,T)$ the expression (4.12), we obtain that

$$\frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon(T)|^2 + \int_0^T E(u^\varepsilon(s))\,ds + \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon |u^\varepsilon|^2 = \frac{1}{2\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 + \int_0^T\int_\Omega f_\varepsilon u^\varepsilon + \frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon$$

where $E(u^\varepsilon) = \int_\Omega |\nabla u^\varepsilon|^2 + \lambda \int_\Omega |u^\varepsilon|^2$.

Now observe that from Corollary 3.2 we have

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \to \int_\Gamma |u_0|^2,$$

while

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon(T)|^2 \to \int_\Gamma |u^0(T)|^2.$$

For this last statement, observe that, from (4.28), $\|u^\varepsilon(T)\|^2_{H^1(\Omega)} \leq C$, while the convergence in Step 2, he have

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} u^\varepsilon(T) \to u^0_{|\Gamma}(T) \quad \text{strongly in } H^{-1}(\Omega).$$

Hence, the arguments in Corollary 3.2 conclude.

Next,with $\rho > 2(N-1)$ from Lemma 3.8 we get (3.24) and

$$\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon} V_\varepsilon |u^\varepsilon|^2 \to \int_0^T\int_\Gamma V|u^0|^2.$$

Therefore, passing to the limit in the energy equality above, we obtain that

$$\frac{1}{2}\int_\Gamma |u^0(T)|^2 + \lim_{\varepsilon\to 0}\Big(\int_0^T E(u^\varepsilon(s))ds\Big) + \int_0^T\int_\Gamma V|u^0|^2 = \frac{1}{2}\int_\Gamma |u_0|^2 + \int_0^T\int_\Omega f u^0 + \int_0^T\int_\Gamma g u^0. \tag{4.29}$$

On the other hand, multiplying (1.2) by u^0 in $L^2(\Omega)$ and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma |u^0|^2 + E(u^0) + \int_\Gamma V|u^0|^2 = \int_\Omega f u^0 + \int_\Gamma g u^0$$

with $E(u^0) = \int_\Omega |\nabla u^0|^2 + \lambda \int_\Omega |u^0|^2$. Integrating in $t \in (0,T)$ the expression above, we obtain that

$$\frac{1}{2}\int_\Gamma |u^0(T)|^2 + \int_0^T E(u^0) + \int_0^T\int_\Gamma V|u^0|^2 = \frac{1}{2}\int_\Gamma |u_0|^2 + \int_0^T\int_\Omega f u^0 + \int_0^T\int_\Gamma g u^0.$$

and comparing with (4.29) we conclude that

$$\int_0^T E(u^0(s))ds = \lim_{\varepsilon\to 0}\int_0^T E(u^\varepsilon(s))ds$$

and we get that u^ε converges to u^0 in $L^2((0,T), H^1(\Omega))$. $\square$

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14. DYNAMIC BOUNDARY CONDITIONS AS A SINGULAR LIMIT OF PARABOLIC PROBLEMS WITH TERMS CONCENTRATING AT THE BOUNDARY, A. Jimémez-Casa and A. Rodríguez Bernal